Pdf-04 (01) Algebra ; Dr Sahibter

# Maximal subgroup

A normal subgroup $H$ of a group $G$ is said to be maximal if there exists no proper normal subgroup $K$ of $G$ such that $H \subset K \subset G$.

**Theorem.** A normal subgroup $H$ of a group $G$ is maximal iff $\frac{G}{H}$ is simple.

**Proof.** Suppose $H$ be a normal subgroup of a group $G$

"If part" Suppose $H$ is maximal

To prove $\frac{G}{H}$ is simple

Suppose if possible $\frac{G}{H}$ is not simple

$\Rightarrow \frac{G}{H}$ has proper normal subgroups

let $\frac{K}{H}$ be a proper normal subgroup of $\frac{G}{H}$

$\Rightarrow K \Delta G$ such that $H \subset K \subset G$ [$\because \frac{K}{H}$ is proper normal subgroup of $\frac{G}{H}$]

$H \subset K \subset G$ means $H$ is not maximal in $G$

So we can a contradiction [because $H$ is maximal in $G$ we mean $\exists$ no normal subgps b/w $H$ & $G$.]

$\Rightarrow \frac{G}{H}$ must be simple.

"Only if part" Suppose $\frac{G}{H}$ is simple

To prove $H$ is maximal in $G$

Suppose if possible $H$ is not maximal in $G$

$\Rightarrow \exists$ normal subgroup $K$ such that $H \subset K \subset G$.

$\Rightarrow \frac{K}{H} \underset{(proper)}{\Delta} \frac{G}{H}$ [i.e $\frac{K}{H}$ is a proper normal subgroup of $\frac{G}{H}$]

$\Rightarrow \frac{G}{H}$ has proper normal subgroup namely $\frac{K}{H}$

$\Rightarrow \frac{G}{H}$ can not be simple, so we get a contradiction

$\Rightarrow$ $H$ is maximal in $G$.

Pdf-04

② Algebra, Dr Satish Kumar

## Composition series [C.S.]

Let $G$ be a group. Then a finite sequence of its subgroups

$$G = H_1, H_2, H_3, \dots, H_n = \{e\}$$

is called a composition series for $G$ if except $H_1$

$H_2 \underset{max}{\Delta} H_1$ i.e. $H_2$ is maximal normal subgroup of $H_1$

$H_3 \underset{max}{\Delta} H_2$ i.e. $H_3$ is maximal normal subgroup of $H_2$

$H_4 \underset{max}{\Delta} H_3$ i.e. $H_4$ is maximal normal subgroup of $H_3$

⋮

$H_i \underset{max}{\Delta} H_{i-1}$, $\quad i = 2, 3, 4, \dots n$.

[i.e $H_n \underset{max}{\Delta} H_{n-1}$] i.e $H_n$ is maximal normal subgroup of $H_{n-1}$

Note. The quotients groups namely

$\frac{H_1}{H_2}, \frac{H_2}{H_3}, \frac{H_3}{H_4}, \dots \frac{H_{n-1}}{H_n}$ are called facter groups. OR composition quotient groups of C.S.

Example. Let $G = P_3 = \{I, (12), (13), (23), (123), (132)\}$
① be a group.

and Let $H = \{I, (123), (132)\}$

then composition series of subgroups ~~are~~ is given by

$$G = H_1, \; H_2 = H = \{I, (123), (132)\}, \; H_3 = \{I\}$$

Here $H_2 \underset{maximal}{\Delta} H_1$ and $H_3 \underset{maximal}{\Delta} H_2$

i.e $H_2$ is maximal normal subgroup of $H_1$

$H_3$ is maximal normal subgroup of $H_2$

Example.
② Let $G$ be a cyclic group of order 6 generated by $a \in G$.

C.S is given by:
(1) $G = H_1 = (a)$, $H_2 = \{e, a^3\}$, $H_3 = \{e\}$

(2) ∴ One more c.s for G

$G=(a)=H_1,\ H_2=\{e,a^2,a^4\},\ H_3=\{e\}$

Here $H_2$ is maximal normal subgroup of $H_1$

$H_3$ is maximal normal subgroup of $H_2$

[i.e $H_1 \supset H_2 \supset H_3$].

Example 03 Let G be a cyclic group generated by a, $a \in G$ such that $o(G)=12$

then

First c.s. $G=H_1=(a)$, $H_2=(a^2)$, $H_3=(a^4)$, $H_4=(e)$

Second c.s $G=H_1=(a)$, $H_2=(a^3)$, $H_3=(a^6)$, $H_4=\{e\}$

Theorem 1. There exist atleast one composition series for every finite group.

Proof. Let G be a finite group

Two cases arise (1) G is simple (2) G is not simple

Case (I) G is simple ⇒ G has two improper normal subgroups of G

namely $G=H_1$, $H_2=\{e\}$

where $H_2$ is maximal normal subgroup of $H_1$.

Case (2). G is not simple

⇒ there exists proper normal subgroup say H of G

(a) If H is maximal in G and $\{e\}$ is maximal in H then required composition series is given by

$G, H, \{e\}$

(b) Suppose H is not maximal in G and $\{e\}$ is maximal in G.

H is not maximal in G ⇒ ∃ proper normal subgroup K such that $G \supset K \supset H$

if K is maximal in G, H is maximal in K then required composition series is

$G, K, H, \{e\}$

(c) Suppose $H$ is maximal in $G$ but $\{e\}$ is not maximal in $H$

Now,

$\{e\}$ is not maximal in $H \Rightarrow \exists$ proper normal subgroup $K_1$ such that $H \supset K_1 \supset \{e\}$

If $K_1$ is maximal in $H$ and $\{e\}$ is maximal in $K_1$ then required composition series is given by

$$G, H, K_1, \{e\}$$

(d) Suppose $H$ is not maximal in $G$, $\{e\}$ is not maximal in $H$

Now,

$H$ is not maximal in $G \Rightarrow \exists$ proper normal subgroup $K_2$ such that $G \supset K_2 \supset H$

$\{e\}$ is not maximal in $H \Rightarrow \exists$ proper normal subgroup $K_3$ such that $H \supset K_3 \supset \{e\}$

if $K_2$ is maximal in $G$, $H$ is maximal in $K_2$, $K_3$ is maximal in $H$ and $\{e\}$ is maximal in $K_3$ then the required composition series is

$$G, K_2, H, K_3, \{e\}.$$

Since $G$ is finite it means no of subgroups of $G$ will be finite and ultimately we shall reach to composition series. proved.

Theorem 2 Jordan-Holder Theorem

Let G be a finite group with two composition series

$G, H_1, H_2, \ldots, H_n = \{e\}$ ——

and $G, K_1, K_2, \ldots, K_m = \{e\}$ ——

Then $n = m$ and the two corresponding series of composition quotient groups namely

$\frac{G}{H_1}, \frac{H_1}{H_2}, \frac{H_2}{H_3}, \ldots, \frac{H_{n-1}}{H_n}$ ——

and $\frac{G}{K_1}, \frac{K_1}{K_2}, \frac{K_2}{K_3}, \ldots, \frac{K_{m-1}}{K_m}$ ——

are abstractly identical.

Proof. Suppose G be a finite group having two C.S. as

$G, H_1, H_2, \ldots, H_n$ —— (1)

and $G, K_1, K_2, \ldots, K_m$ —— (2)

then to prove (1) $n = m$

(2) $\frac{G}{H_1}, \frac{H_1}{H_2}, \ldots \frac{H_{n-1}}{H_n}$ and $\frac{G}{K_1}, \frac{K_1}{K_2}, \ldots \frac{K_{m-1}}{K_m}$

are abstractly identical.

We shall prove the theorem by induction method

Suppose the theorem is true for all groups whose order are less than $O(G)$. Then we shall prove it is also true for G.

1) if $O(G) = 1$, theorem is obvious.

(2) If $O(G) \geq 2$.

Case I If $H_1 = K_1$ In this case after removing G from (1) and (2), we get the remaining series as

$H_1, H_2, \ldots, H_n$ —— (3)

and $K_1, K_2, \ldots, K_m$ —— (4)

Here $O(H_1) < O(G)$ [$\because$ $H_1$ is a proper subgroup of $G$]

so theorem is true by our supposition

Since $\frac{G}{H_1} = \frac{G}{K_1}$ [$\because H_1 = K_1$]

$\therefore$ corresponding corresponding quotient group are equal.

<u>Case 2</u>, $H_1 \neq K_1$

We know that

$$\frac{H_1 K_1}{H_1} \cong \frac{K_1}{H_1 \cap K_1} \quad \text{——} \quad (5)$$

and $$\frac{H_1 K_1}{K_1} \cong \frac{H_1}{H_1 \cap K_1} \quad \text{——} \quad (6)$$

Here $H_1 K_1$ is a normal subgroup of $G$ containing $H_1$

since $H_1$ is maximal in $G$

$\Rightarrow H_1 K_1 = G$

$\therefore$ (5) & (6) $\Rightarrow$ $\frac{G}{H_1} \cong \frac{K_1}{D}$, $H_1 \cap K_1 = D$; say —— (7)

and $\frac{G}{K_1} \cong \frac{H_1}{D}$ —— (8)

Now, $H_1$ is maximal in $G$ $\Rightarrow$ $\frac{G}{H_1}$ is simple

$\Rightarrow \frac{K_1}{D}$ is simple [$\because \frac{G}{H_1} \cong \frac{K_1}{D}$]

$\Rightarrow D$ is maximal in $K_1$

Similarly $D$ is maximal in $H_1$

Now, <u>consider C.S for D</u>

Let $D, D_1, D_2, \ldots, D_t = \{e\}$ be a C.S. for $D$ —— (9)

then $G, H_1, D, D_1, D_2, \ldots, D_t = \{e\}$ —— (10)

and $G, K_1, D, D_1, D_2, \ldots, D_t = \{e\}$

are two composition series for $G$.

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Let us have composition quotient groups of (9) & (10)

$$\frac{G}{H_1}, \frac{H_1}{D}, \frac{D}{D_1}, \frac{D_1}{D_2}, \cdots, \frac{D_{t-1}}{D_t} \quad \text{--- (11)}$$

and $$\frac{G}{K_1}, \frac{K_1}{D}, \frac{D}{D_1}, \frac{D_1}{D_2}, \cdots, \frac{D_{t-1}}{D_t} \quad \text{--- (12)}$$

The quotient groups are equal in numbers

and $\frac{G}{H_1} \cong \frac{K_1}{D}$

$\frac{G}{K_1} \cong \frac{H_1}{D}$

$\frac{D}{D_1} \cong \frac{D}{D_1}$

$\frac{D_1}{D_2} \cong \frac{D_1}{D_2}$

$\vdots$

$\frac{D_{t-1}}{D_t} \cong \frac{D_{t-1}}{D_t}$ are each isomorphic separately.

Now (1) and (9) are two c. s for G and each having $H_1$ in second place

$\Rightarrow$ by Case 1, the quotient groups defined by (1) & (9) may be put into 1-1- correspondence so that corresponding quotient groups are isomorphic

Similarly (2) & (10) are two c. s for G and each having $K_1$ in second place

& by case I, the quotient groups defined by (2) & (10) may be put into one-one correspondence so that corresponding quotient groups are isomorphic

Hence $\frac{G}{H_1}, \frac{H_1}{H_2}, \cdots \frac{H_{n-1}}{H_n}$ and $\frac{G}{K_1}, \frac{K_1}{K_2}, \cdots \frac{K_{m-1}}{K_m}$ are equal.

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## Solvable groups

A group $G$ is said to be solvable if we can find a finite chain of subgroups

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k = \{e\}$$

such that

(1) Each $N_i \trianglelefteq N_{i-1}$, $i = 1, 2, \ldots, k$.

(2) Each quotient gp $\frac{N_{i-1}}{N_i}$ is abelian, $i = 1, 2, \ldots, k$.

Note (1) Each $N_i \trianglelefteq N_{i-1} \Rightarrow N_1 \trianglelefteq N_0, N_2 \trianglelefteq N_1, N_3 \trianglelefteq N_2, \cdots N_k \trianglelefteq N_{k-1}$

(2) Each quotient gp is $\frac{N_i}{N_{i-1}}$ is abelian we mean

Each $\frac{N_0}{N_1}, \frac{N_1}{N_2}, \frac{N_2}{N_3}, \cdots, \frac{N_{k-1}}{N_k}$ is abelian

Ex. (1) Show every abelian group is solvable

Sol Let $G$ be an abelian group

Then we have solvable series for $G$ as

$G = N_0$ and $N_1 = \{e\}$

clearly $N_1 \trianglelefteq N_0$ $\quad [\because x e x^{-1} \in N_1, x \in N_0]$

and $\frac{N_0}{N_1}$ is abelian $\left\{ \frac{N_0}{N_1} = \{N_1 . x, \forall x \in N_0\} \right.$
$= N_0$ (since $N_1 = \{e\}$ & $ex = x$ )

Ex (2). Show that symmetric group $P_3$ of degree 3 is solvable.
if $S = (1, 2, 3)$

Sol. Let $P_3 = \{I, (12), (13), (23), (123), (132)\}$

Let $A_3 = \{I, (123), (132)\}$ be alternating group

then solvable series for $P_3$ is

$P_3 = N_0$, $\quad N_1 = A_3$, $\quad N_2 = \{e\} \Rightarrow$ $N_1 \trianglelefteq N_0$, $N_2 \trianglelefteq N_1$.

Also $\frac{N_0}{N_1}$ is abelian, $\frac{N_1}{N_2}$ is abelian $\quad \frac{N_1}{N_2} = \{N_2 x, \forall x \in A_3\}$
$= A_3$ as $N_2 = \{e\}$

so, $P_3$ is solvable.

(3) if $S = (1, 2, 3, 4)$ then $P_4$ is solvable

& solvable series is

$P_4 = N_0$, $N_1 = A_4$, $N_2 = V_4 = \{I, (12)(34), (13)(24), (14)(23)\}$, $N_3 = \{e\}$

$\Rightarrow N_1 \trianglelefteq N_0, N_2 \trianglelefteq N_1, N_3 \trianglelefteq N_2$

& $\frac{N_0}{N_1}, \frac{N_1}{N_2}, \frac{N_2}{N_3}$, each is abelian.

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Prove that a subgroup of a solvable group is solvable.

Proof. Suppose $H$ be any subgroup of a solvable group $G$.

To prove $H$ is solvable

It is given $G$ is solvable $\Rightarrow$ $\exists$ solvable series namely

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = \{e\}.$$

We claim that

$$H = H_0 \supseteq (H \cap G_1) \supseteq (H \cap G_2) \supseteq (H \cap G_3) \supseteq \cdots \supseteq (H \cap G_n) \quad \text{--- (1)}$$

Let $H_i = H \cap G_i$ then

(1) $\Rightarrow$

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots \supseteq H_n = \{e\}$$

Clearly

$H_1 \trianglelefteq H_0$ $[\because H_0 = H,\ H_1 = H \cap G_1$ and $G_1 \trianglelefteq G,\ H \subseteq G]$

$H_2 \trianglelefteq H_1$ $[H_2 = H \cap G_2,\ H_1 = H \cap G_1]$

$\vdots$

$H_n \trianglelefteq H_{n-1}$ $[H_n = H \cap G_n,\ H_{n-1} = H \cap G_{n-1}]$

To prove each

$$\frac{H_0}{H_1}, \frac{H_1}{H_2}, \frac{H_2}{H_3}, \cdots \frac{H_{n-1}}{H_n} \text{ is abelian}$$

Define a mapping

$$\phi : H_i \longrightarrow \frac{G_i}{G_{i+1}}, \quad i = 0, 1, 2, \cdots, (n-1).$$

$$\phi(x) = x G_{i+1}, \quad \forall\, x \in H_i$$

Here $x \in H_i \Rightarrow x \in G_i$ as $H_i \subseteq G_i$

Clearly $\phi$ is well defined

$$\phi(xy) = xy\, G_{i+1}$$
$$= x G_{i+1}\, y G_{i+1} \quad [\because \cancel{Ha =} abH = aH\, bH,\ a, b \in G,\ H \trianglelefteq G]$$
$$= \phi(x)\, \phi(y)$$

$\Rightarrow$ $\phi$ is homomorphism $\Rightarrow$ $\dfrac{G_i}{G_{i+1}}$ is homomorphic image of $H_i$.

To find Kernel of $\phi$

Let $x \in$ Kernel of $\phi$ $\Leftrightarrow$ $\phi(x) = G_{i+1}$

$\Leftrightarrow x G_{i+1} = G_{i+1}$

$\Leftrightarrow x \in G_{i+1}$

$\Leftrightarrow x \in H \cap G_{i+1}$

$\Leftrightarrow x \in H_{i+1}$

$x \in$ Kernel of $\phi \Leftrightarrow x \in H_{i+1}$

$\Rightarrow$ Kernel of $\phi = H_{i+1}$

So by first fundamental theorem on groups, we have

$$\frac{H_i}{H_{i+1}} \cong \frac{G_i}{G_{i+1}}$$

$\Rightarrow$ Each $\frac{H_i}{H_{i+1}}$ is abelian as each $\frac{G_i}{G_{i+1}}$ is abelian

Hence $H$ is solvable

Ex If Let $H$ be a normal subgroup of a group $G$ such that both $N$ and $\frac{G}{N}$ are solvable, prove $G$ is solvable.

Solution. ~~Let~~ Let $H$ be a normal subgroup of a group $G$

Let $H$ is solvable

$\frac{G}{H}$ is solvable

To prove $G$ is solvable

$\frac{G}{N}$ is solvable $\Rightarrow \exists$ solvable series namely

$$\frac{G}{N} = \frac{G_0}{N} \supseteq \frac{G_1}{N} \supseteq \frac{G_2}{N} \supseteq \text{---} \supseteq \frac{G_m}{N} = \{N\}$$

where
$$\left.\begin{aligned} &\frac{G_1}{N} \triangleright \frac{G_0}{N} \Rightarrow G_1 \triangleright G_0 \text{ containing } N \\ &\frac{G_2}{N} \triangleright \frac{G_1}{N} \Rightarrow G_2 \triangleright G_1 \text{ containing } N \\ &\quad \equiv \\ &\frac{G_m}{N} \triangleright \frac{G_{m-1}}{N} \Rightarrow G_m \triangleright G_{m-1} \text{ containing } N \end{aligned}\right\} \text{—(1)}$$

and each $G$

(1) $\Rightarrow \frac{G_{i+1}}{N} \triangleright \frac{G_i}{N}$, $i = 0, 1, 2, \cdots (m-1)$. $\Rightarrow G_{i+1} \triangleright G_i$ st $G_i \supseteq N$

Also
$$\frac{\frac{G_i}{N}}{\frac{G_{i+1}}{N}} \cong \frac{G_i}{G_{i+1}} \qquad \left[\because \frac{\frac{G}{H}}{\frac{K}{H}} \cong \frac{G}{K}\right]$$

It is given $\frac{G}{N}$ is solvable

$\Rightarrow$ Each $\dfrac{G_i/N}{G_{i+1}/N}$ is solvable because each subgroup of a solvable group is solvable

$\Rightarrow$ Each $\frac{G_i}{G_{i+1}}$ is ~~solvable~~ abelian $\left[\because \dfrac{G_i/N}{G_{i+1}/N} \cong \dfrac{G_i}{G_{i+1}}\right]$ — (2)

Also
$$\frac{G_m}{N} = \{Ne \mid e \in G_m\}$$
$$\frac{G_m}{N} = N \quad [\because G_m = \{e\}]$$

It is also given that $N$ is solvable $\Rightarrow \exists$ its solvable series namely

$$N = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_t = \{e\}$$

Now, we can write

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{m-1} \supseteq N_0 \supseteq N_1 \supseteq \cdots \supseteq N_t = \{e\} \quad [\because N = N_0]$$

which is a solvable series for $G$ By [eqn's (1) & (2)]

$\Rightarrow G$ is solvable.

## Commutator subgroup of a group

Let $G$ be a group and $a, b \in G$ then the element $(aba^{-1}b^{-1})$ is called the commutator of $(a, b)$

Let $U = \{aba^{-1}b^{-1} : a, b \in G\}$

If $G'$ is subgroup of $G$ generated by $U$ then $G'$ is called commutator subgroup of $G$

i.e. if $G' = (U)$, then $G'$ is called commutator subgroup of $G$

i.e. $G' = \{aba^{-1}b^{-1} \mid a, b \in G\}$

Note: We can also write

$$U = \{a^{-1}b^{-1}(a^{-1})^{-1}(b^{-1})^{-1} \mid a, b \in G\}$$
$$U = \{a^{-1}b^{-1}ab \mid a, b \in G\}$$

**Theorem 1.** Let $G'$ be the commutator subgroup of a group $G$. Then $G$ is abelian if and only if $G' = \{e\}$, $e$ being the identity element of $G$.

**Proof.** Let $G'$ be the commutator subgroup of a group $G$

then $G' = \{ aba^{-1}b^{-1} \mid a, b \in G \}$

"if part" Suppose $G$ is abelian

To prove $G' = \{e\}$

Let $a, b \in G$ then

$aba^{-1}b^{-1} \in G' \Rightarrow (aba^{-1}b^{-1}) = a(ba^{-1})b^{-1}$ (Associativity)

$= a(a^{-1}b)b^{-1}$ [$G$ is abelian

$= (aa^{-1})(bb^{-1})$ (Associativity)

$= e.e$

$= e$

$\Rightarrow G' = \{e\}$

"only if part" Suppose $G' = \{e\}$

To prove $G$ is abelian

Let $aba^{-1}b^{-1} \in G' \Rightarrow ab(ba)^{-1} \in G'$

$\Rightarrow ab(ba)^{-1} = \{e\}$ [$\because G' = \{e\}$]

$\Rightarrow ab = ba$

$\Rightarrow G$ is abelian.

**Theorem 2.** Let $G$ be a group. Let $G'$ be the commutator subgroup of $G$ then

(1) $G'$ is normal in $G$ (2) $\frac{G}{G'}$ is abelian

(3) If $N \trianglelefteq G$ then $\frac{G}{N}$ is abelian iff $G' \subseteq N$

(4) If $H$ is a subgroup of $G$ such that $G' \subseteq H$, then $H$ is a normal subgroup of $G$.

**Proof.** Let $G'$ be the commutator subgroup of $G$

(1) To prove $G' \trianglelefteq G$

Let $x \in G, c \in G'$

then $xcx^{-1} = xcx^{-1}c^{-1}.c \in G'$ [$\because xcx^{-1}c^{-1} \in G', c \in G'$ $\Rightarrow xcx^{-1}c^{-1}.c \in G'$]

$\Rightarrow xcx^{-1} \in G' \Rightarrow G' \trianglelefteq G$.

(2) To prove $\frac{G}{G'}$ is abelian

Let $aba^{-1}b^{-1} \in G' \Rightarrow ab(ba)^{-1} \in G'$

$\Rightarrow G'ab = G'ba$ $[\because Ha = Hb \Leftrightarrow ab^{-1} \in H]$

$\Rightarrow G'aG'b = G'bG'a$ $[G' \trianglelefteq G]$

$\Rightarrow \frac{G}{G'}$ is abelian

(3) If $N \trianglelefteq G$ then $\frac{G}{N}$ is abelian iff $G' \subseteq N$

Suppose $N \trianglelefteq G$

"If part" Suppose $\frac{G}{N}$ is abelian

To prove $G' \subseteq N$

It is given $\frac{G}{N}$ is abelian $\Rightarrow NaNb = NbNa$ [where $a, b \in G$]

$\Rightarrow Nab = Nba$ $[N \trianglelefteq G]$

$\Rightarrow ab(ba)^{-1} \in N$

Thus $\Rightarrow aba^{-1}b^{-1} \in N$

$aba^{-1}b^{-1} \in G' \Rightarrow aba^{-1}b^{-1} \in N$

$\Rightarrow G' \subseteq N$.

"Only if part" Suppose $G' \subseteq N$

To prove $\frac{G}{N}$ is abelian

Let $aba^{-1}b^{-1} \in G' \Rightarrow aba^{-1}b^{-1} \in N$ $[G' \subseteq N]$

$\Rightarrow ab(ba)^{-1} \in N$

$\Rightarrow Nab = Nba$

$\Rightarrow NaNb = NbNa$ $[\because N \trianglelefteq G]$

$\Rightarrow \frac{G}{N}$ is abelian

(4) If $H$ is a subgroup of $G$ such that $G' \subseteq H$

To prove $H \trianglelefteq G$

Let $x \in G, h \in H$

$xhx^{-1} = \underline{xhx^{-1}h^{-1}} \cdot \underline{h} \in H$ $[\because xhx^{-1}h^{-1} \in G'$ $\Rightarrow xhx^{-1}h^{-1} \in H$ $(G' \subseteq H)$ also $h \in H$ $\Rightarrow xhx^{-1}h^{-1} \cdot h \in H]$

$\Rightarrow xhx^{-1} \in H$

$\Rightarrow H \trianglelefteq G$.

**Theorem 03.** A group $G$ is solvable iff $G^{(k)} = (e)$ for some integer $k$.

**Proof.** Let $G$ be a group

"if part" Suppose $G^{k} = (e)$

To prove $G$ is solvable

Given $G^{k} = (e)$

Let $G = N_0$, $N_1 = G'$, $N_2 = G^{(2)}$, $N_3 = G^{(3)} \cdots N_k = G^{(k)} = \{e\}$

$\Rightarrow G = N_0 \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots \supseteq N_{k-1} \supseteq N_k = \{e\}$ —(1)

Let us claim that (1) is solvable series for $G$.

We $N_1 \triangleleft N_0$ because $G' \triangleleft G$, [Here $G' = N_1$ and $N_0 = G$]

$N_2 \triangleleft N_1$ because $G^{(2)} \triangleleft G'$ [Here $N_2 = G^{(2)}$ and $N_1 = G^{(1)}$ and we know $G^{(2)} \triangleleft G'$]

$\vdots$

$N_k \triangleleft N_{k-1}$ because $G^{(k)} \triangleleft G^{(k-1)}$ [Here $G^{(k)} = N_k$ and $G^{(k-1)} = N_{k-1}$]

Also $\frac{N_0}{N_1}$ is abelian as $\frac{G}{G'}$ is abelian

$\frac{N_1}{N_2}$ is abelian as $\frac{G'}{G^{(2)}}$ is abelian ($G^{(2)} = (G')'$)

$\vdots$

$\frac{N_{k-1}}{N_k}$ is abelian as $\frac{G^{(k-1)}}{G^{(k)}} = \frac{G^{(k-1)}}{G^{(k-1)\prime}}$, is abelian

Therefore, $G$ is solvable

• "Only if part" Suppose $G$ is solvable

To prove $G^{(k)} = e$

given $G$ is solvable $\Rightarrow G$ has solvable series as

$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_{k-1} \supseteq N_k = \{e\}$

where Each $N_i \triangleleft N_{i-1}$, $i = 1, 2, \ldots, k$.

and Each $\frac{N_{i-1}}{N_i}$ is abelian —(1)

We know that if $N \triangleleft G$ then $\frac{G}{N}$ is abelian $\Rightarrow G' \subseteq N$

$\therefore$ (1) $\Rightarrow$ Each $N_{i-1}' \subseteq N_i$, $i = 1, 2, 3 \ldots k$.

$\Rightarrow$ $N_1 \supseteq N_0' = G'$

$N_2 \supseteq N_1' = G^{(2)}$ [$\because N_1 = G^{(1)} \Rightarrow N_1' = G^{(2)}$]

$\vdots$

$N_k \supseteq N_{k-1}' = G^{(k)}$ $\Rightarrow G^{(k)} \subseteq N_k \Rightarrow G^{(k)} \subseteq (e)$ [$\because N_k = (e)$]

But $e \subseteq G^{(k)}$

$\Rightarrow G^{(k)} = (e)$, proved.